# Wave Equation from Maxwell's Equations

Maxwell's equations for a region with no charge or current are, in differential form:

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{3}$$

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

Here I have assumed that the the charge density $\rho$ and current density $\mathbf{J}$ are zero, and that the electric displacement vector can be expressed as $\mathbf{D} = \epsilon \mathbf{E}$ and the magnetic flux can be expressed as $\mathbf{H} = \mu \mathbf{B}$, which are common assumptions. Thus, the speed $c$ in the equations above refer to the speed of light in the particular medium $c = 1/\sqrt{\epsilon\mu}$, and in the case of a vacuum, $c$ is the standard speed of light in vacuum.

In this problem I will focus exclusively on the magnetic field components. Since the above equations are coupled, it suffices to solve either for the electric field $\mathbf{E}$ or magnetic field $\mathbf{B}$, and then the other vector can be inferred. This leads to a set of vector wave equation:

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{5}$$

which holds for each component of the vector $\mathbf{B}$. The above equation can be verified by using the vector differential identity,

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \tag{6}$$

When taking $\nabla^2 \mathbf{B}$, one can use $\nabla \cdot \mathbf{B} = 0$ to eliminate the first term on the right hand side, leaving only the second curl-of-curl term. The interior curl can be found by again using Maxwell's equations, leaving

$$\nabla^2 \mathbf{B} = -\nabla \times \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right). \tag{7}$$

Upon exchanging the order of differentiation on the right hand side, and using Maxwell's equations one last time for $\nabla \times \mathbf{E}$, I verify the the wave equation (5) is correct.


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